DeepBayes Summer School - Theoretical Assignments

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April 15, 2019

1 Problem 1

Knowing that the sum of n i.i.d Bernoulli random variables $X_i \sim Bern(p)$, $Z = \sum_{i=1}^{n} X_i$ is a Binomially distributed with $Z \sim Bin(n, p)$, we write the model as

$$\begin{cases} \xi \,|\, \lambda \sim \text{Pois}(\lambda),\ \xi \in \{0, 1, \dots\} \\ \eta \,|\, \xi, \lambda \sim \text{Bin}(\xi, p), \eta \in \{0, 1, \dots \xi\} \end{cases} \tag{1}$$

Then, we compute the pdf $f(\eta|\lambda)$ as

$$\begin{aligned} f(\eta|\lambda) &= \sum_{\xi=\eta}^{\infty} \left[\binom{\xi}{\eta} p^{\eta} (1-p)^{\xi-\eta} \right] \left[\frac{\lambda^{\xi} e^{-\lambda}}{\xi!} \right] && (2) \\ &= \sum_{\xi=\eta}^{\infty} \frac{\xi!}{\eta!(\xi-\eta)!} p^{\eta} (1-p)^{\xi-\eta} \frac{\lambda^{\xi} e^{-\lambda}}{\xi!} && (3) \\ &= \frac{p^{\eta}}{\eta!} \sum_{\xi=\eta}^{\infty} \frac{1}{(\xi-\eta)!} (1-p)^{\xi-\eta} \lambda^{\xi} e^{-\lambda}. && (4) \end{aligned}$$

Doing the following change of variables in the summation: $k = \xi - \eta$; we obtain:

$$\begin{aligned} f(\eta|\lambda) &= \frac{p^{\eta}}{\eta!} \sum_{k=\xi-\eta=0}^{\infty} \frac{(1-p)^k}{k!} \lambda^{k+\eta} e^{-\lambda} && (5) \\ &= \frac{(p\lambda)^{\eta}}{\eta!} \sum_{k=0}^{\infty} \frac{(1-p)^k}{k!} \lambda^{k} e^{-\lambda} && (6) \\ &= \frac{(p\lambda)^{\eta}}{\eta!} \sum_{k=0}^{\infty} \frac{(1-p)^k}{k!} \lambda^{k} e^{-\lambda} \times \frac{e^{-\lambda p}}{e^{-\lambda p}} && (7) \\ &= \frac{(p\lambda)^{\eta} e^{-\lambda p}}{\eta!} \sum_{k=0}^{\infty} \frac{(1-p)^k}{k!} \lambda^{k} e^{-\lambda(1-p)} && (8) \\ &= \frac{(p\lambda)^{\eta} e^{-\lambda p}}{\eta!} \underbrace{\sum_{k=0}^{\infty} \text{Pois}(k; \lambda(1-p))}_{=1} && (9) \\ &= \frac{(p\lambda)^{\eta} e^{-\lambda p}}{\eta!} && (10) \\ \eta|\lambda &\sim \text{Pois}(p\lambda) && (11) \end{aligned}$$

2 Problem 2

We start by using the Bayes rule of probability to the conditional distribution $p(\text{kind}|t=10)$, which gives

$$p(\text{kind}|t=10) = \frac{p(t=10|\text{kind})p(\text{kind})}{p(t=10|\text{kind})p(\text{kind}) + p(t=10|\text{strict})p(\text{strict})}. \tag{12}$$

Knowing that $p(\text{strict}) = p(\text{kind}) = 0.5$, we can write the above formula as

$$p(\text{kind}|t=10) = \frac{p(t=10|\text{kind})}{p(t=10|\text{kind}) + p(t=10|\text{strict})} \underbrace{\frac{p(\text{strict})}{p(\text{strict})}}_{=1} \tag{13}$$

$$= \left(1 + \frac{p(t=10|\text{strict})}{p(t=10|\text{kind})}\right)^{-1}. \tag{14}$$

Further knowing the model for each reviewer to be a Gaussian (with different parameters), we do

$$p(\text{kind}|t=10) = \left[1 + \frac{\left(2\pi\sigma_1^2\right)^{-1/2} \exp\left[-\frac{1}{2\sigma_1^2}\left(t-\mu_1\right)^2\right]}{\left(2\pi\sigma_2^2\right)^{-1/2} \exp\left[-\frac{1}{2\sigma_2^2}\left(t-\mu_2\right)^2\right]}\right]^{-1} \tag{15}$$

$$= \left[1 + \frac{\sigma_2}{\sigma_1} \exp\left\{-\frac{1}{2}\left[\frac{1}{\sigma_1^2}\left(t-\mu_1\right)^2 - \frac{1}{\sigma_2^2}\left(t-\mu_2\right)^2\right]\right\}\right]^{-1}, \tag{16}$$

and now substituting the value given for μ_1, σ_1, μ_2, and σ_2, we obtain

$$p(\text{kind}|t=10) = \left[1 + \frac{5}{10} \exp\left\{-\frac{1}{2}\left[\frac{1}{100}\left(10-30\right)^2 - \frac{1}{25}\left(10-20\right)^2\right]\right\}\right]^{-1} \tag{17}$$

$$= \left[1 + \frac{1}{2} \exp\left\{-\frac{1}{2}(4-4)\right\}\right]^{-1} \tag{18}$$

$$= \frac{2}{3} \tag{19}$$